# A STUDY OF INSTANTANEOUS AND NON-INSTANTANEOUS IMPULSIVE DIFFERENTIAL EQUATIONS WITH APPLICATIONS IN CONTROL PROBLEMS

*A THESIS*

*submitted by*

**AVADHESH KUMAR**

*for the award of the degree*

*of*

**DOCTOR OF PHILOSOPHY**

**SCHOOL OF BASIC SCIENCES**

**INDIAN INSTITUTE OF TECHNOLOGY MANDI**

**KAMAND-175005, INDIA**

**JULY, 2018**

*Dedicated*

*To*

*My dear parents and beloved wife Shilpy,*

*their love, care, patience and dedication have*

*been a constant source of inspiration for me*

# Declaration by the Research Advisor

I hereby certify that the entire work in this thesis has been carried out by **Mr. Avadhesh Kumar (Enrollment No: D14018)**, under my supervision in the **School of Basic Sciences**, Indian Institute of Technology Mandi and that no part of it has been submitted elsewhere for any degree or diploma.

Signature:

Name of the Guide: Dr. Muslim

Date:

# Declaration by the Research Scholar

I hereby declare that the entire work embodied in this thesis is the result of investigations carried out by me in the **School of Basic Sciences**, Indian Institute of Technology Mandi, under the supervision of **Dr. Muslim** and that it has not been submitted elsewhere for any degree or diploma. In keeping with the general practice, due acknowledgments have been made wherever the work described is based on the finding of other investigators.

Place: Kamand, 175 005

Date:

Signature:

Avadhesh Kumar

# Acknowledgements

*"Gratitude makes sense of our past, brings peace*
*for today, and creates a vision for tomorrow"*

— Melody Beattie

First and foremost, I express my sincere gratitude to my Ph.D. supervisor Dr. Muslim, School of Basic Sciences, Indian Institute of Technology Mandi, India. His great enthusiasm and knowledge introduced me to the interesting and challenging field of infinite dimensional differential equations. I appreciate his support and help during the entire research work and the way he has guided me to become an independent researcher.

I am also grateful to the other members of my doctoral committee: Dr. Syed Abbas, Dr. Manoj Thakur, Dr. Rajendra K. Ray and Dr. Tushar Jain for their valuable suggestions. I acknowledge Prof. Michal Fečkan from Slovakia, Prof. R. P. Agarwal from USA and Prof. R. Sakthivel from Coimbatore, India for their collaboration.

Furthermore, I appreciate the support and help given by the Director of IIT Mandi, Prof. Timothy A. Gonsalves. Also, I would appreciate the support and help given by staff members of IIT Mandi. I would like to extend my thanks to the colleagues in our research group, Rajesh, Vipin, and Shekhar, Soniya. Special thanks to Harmanpreet, Manoj, Jitendra, Ankit, Harivansh, Swati, Subit, Atendra, Pravat, Nishit, Vikas, Haity and Sudeb for their friendship and great memories.

I take this opportunity to thank my well-wishers, Sumit Sharma, Vishal Mishra, Devendra Pratap Singh, Ajay Kumar and Rashmi Savita for their encouragement and stimulation that they provided during the entire tenure of my research work.

Financial support from MHRD is also acknowledged.

I wish to express my heartfelt gratitude and thanks to my parents Mr. Amar Singh and Mrs. Lalita Devi and all my siblings especially my eldest brother Mr. Jitendra Kumar Singh and sister Mrs. Upma Verma for their love and encouragement to boost my moral and confidence during the entire period of this work. I do believe that without the consistent help of my parents and ardent support of my siblings, this work probably would have not seen the ray of the day.

Finally, thanks to Almighty for the kind blessings.

**(Avadhesh Kumar)**

# ABSTRACT

**Keywords**: *Second order nonlinear impulsive systems, Neutral differential equations, Non-instantaneous and instantaneous impulses, Deviated argument, Banach fixed point theorem, Mild periodic solutions, Poincaré operator, Evolution operator, Schauder fixed point theorem, Fractional cosine family, Exact controllability, Trajectory controllability.*

The present research work deals with the investigation of various kinds of impulsive differential models with instantaneous as well as non-instantaneous impulses in abstract spaces. This work provides insight into the different types of controllability, existence and stability of solutions of various second order impulsive models with and without deviated argument. We establish necessary and sufficient conditions for existence, uniqueness and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses. Periodicity is also investigated of an impulsive model described by second order nonlinear differential equations with non-instantaneous impulses. Furthermore, we obtain some significant results on various kinds of controllability for the second order evolution systems with instantaneous impulses and deviated argument. Also, the existence of solution and controllability to second order neutral differential equation with non-instantaneous impulses have been investigated. Finally, the controllability of fractional differential equation of order $\alpha \in (1, 2]$ with non-instantaneous impulses has been established, which has not been proposed so far to the best of our knowledge. This work leads to the motivation for the study of total controllability of the given problem which is a stronger notion of exact controllability. Several examples have been provided in order to make our theoretical analysis more concrete. Also, some numerical results have been carried out to support the analytical findings. In this work, the main used techniques are Banach fixed point theorem, strongly continuous cosine family of bounded linear operators, fractional cosine family, Poincaré operator and evolution operator.

# Contents

**Acknowledgement** **i**

**Abstract** **iv**

**List of Figures** **ix**

**1 Introduction and preliminaries** **1**

1.1 Introduction . . . . . 1

1.1.1 Control Theory . . . . . 1

1.1.2 Controllability . . . . . 2

1.1.3 Stability . . . . . 5

1.1.4 Some Special Kind of Differential Equations . . . . . 6

1.1.5 Impulsive Differential Equations . . . . . 7

1.1.6 Abstract Formulation . . . . . 9

1.2 Literature Survey . . . . . 13

1.3 Preliminaries . . . . . 18

1.3.1 Cosine Family Theory . . . . . 19

1.3.2 Fractional Calculus . . . . . 23

1.3.3 Fractional Order Abstract Differential Equations . . . . . 26

1.4 An Outline of Dissertation . . . . . 29

**2 Existence, uniqueness and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses** **32**

2.1 Introduction . . . . . 32

2.2 Preliminaries and Assumptions . . . . . 35
2.3 Existence and Uniqueness Result . . . . . 37
2.4 Nonlocal Problem . . . . . 43
2.5 Ulam's Type Stability . . . . . 46
2.6 Applications . . . . . 53
2.7 Conclusion . . . . . 56

**3 Periodic solutions to second order nonlinear differential equations with non-instantaneous impulses 57**

3.1 Introduction . . . . . 57
3.2 Preliminaries and Assumptions . . . . . 59
3.3 Existence and Uniqueness of Periodic Solutions . . . . . 61
3.4 Application . . . . . 68
3.5 Conclusion . . . . . 69

**4 Exact and trajectory controllability of second order nonlinear impulsive systems with deviated argument 70**

4.1 Introduction . . . . . 70
4.2 Preliminaries and Assumptions . . . . . 71
4.3 Exact Controllability . . . . . 73
4.4 Controllability of Integro-differential Equation . . . . . 76
4.5 Nonlocal Problems . . . . . 77
4.6 Trajectory Controllability . . . . . 79
4.7 Application . . . . . 81
4.8 Conclusion . . . . . 82

**5 Exact and trajectory controllability of second order evolution systems with impulses and deviated arguments 83**

5.1 Introduction . . . . . 83
5.2 Preliminaries and Assumptions . . . . . 84
5.3 Exact Controllability . . . . . 89

5.4 Trajectory Controllability . . . . . 92
5.5 Applications . . . . . 96
5.6 Conclusion . . . . . 102

**6 Existence and controllability results to second order neutral differential equation with non-instantaneous impulses 103**

6.1 Introduction . . . . . 103
6.2 Preliminaries and Assumptions . . . . . 104
6.3 Existence and Uniqueness Result . . . . . 107
6.4 Controllability Results . . . . . 116
6.5 Trajectory Controllability . . . . . 119
6.6 Application . . . . . 121
6.7 Conclusion . . . . . 123

**7 Controllability of fractional differential equation of order $\alpha \in (1,2]$ with non-instantaneous impulses 124**

7.1 Introduction . . . . . 124
7.2 Preliminaries and Assumptions . . . . . 126
7.3 Exact Controllability . . . . . 129
7.4 Integro-differential Equation . . . . . 134
7.5 Applications . . . . . 137
7.6 Conclusion . . . . . 142

**List of Publications 143**

**Bibliography 146**