STUDY OF DETERMINISTIC AND STOCHASTIC DIFFERENTIAL EQUATIONS WITH APPLICATIONS IN CONTROL PROBLEMS

A THESIS

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Dedicated

To

My dear parents and brother Prashant

Declaration by the Research Advisors

We hereby certify that the entire work in this thesis has been carried out by **Mr. Rajesh Dhayal (Enrollment No: D15026)** under our supervision in the **School of Basic Sciences**, Indian Institute of Technology Mandi and that no part of it has been submitted elsewhere for any degree or diploma.

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Declaration by the Research Scholar

I hereby declare that the entire work embodied in this thesis is the result of investigations carried out by me in the **School of Basic Sciences**, Indian Institute of Technology Mandi, under the supervision of **Dr. Muslim** (Guide) and **Dr. Syed Abbas** (Co-Guide) and that it has not been submitted elsewhere for any degree or diploma. In keeping with the general practice, due acknowledgments have been made wherever the work described is based on the finding of other investigators.

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"Gratitude is born in hearts that take time
to count up past mercies"

– Charles E. Jefferson

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ABSTRACT

Keywords: *Differential equations of integer and non-integer order, Stochastic differential equations, Mild solutions, State-dependent delay, Non-instantaneous impulses, Semigroup theory, q-resolvent family, Banach fixed point theorem, Krasnoselskii's fixed point theorem, Schauder's fixed point theorem, Existence, Stability, Controllability, Existence of optimal controls, Optimal pair, Approximation results.*

The present research work deals with the investigation of various kinds of deterministic and stochastic differential equations in finite as well as in infinite-dimensional spaces. This work provides insight into the different types of controllability, existence, uniqueness, stability and existence of optimal controls for non-instantaneous impulsive differential equations of order one, two and non-integer. We establish the controllability of non-autonomous nonlinear differential equations with non-instantaneous impulses in the space $\mathbb{R}^n$ by using a new piecewise control function. Also, some significant results on various kinds of controllability for fractional differential equation with non-instantaneous impulses and state-dependent delay have been investigated. Further, we establish necessary and sufficient conditions for the existence, uniqueness, stability and controllability of non-instantaneous impulsive stochastic differential equations driven by mixed fractional Brownian motion with Hurst parameter $\mathcal{H} \in (1/2, 1)$. Next, we study the approximate controllability for a class of non-instantaneous impulsive fractional stochastic differential equation driven by fractional Brownian motion in a Hilbert space. The existence of mild solutions and optimal controls for a new class of second-order stochastic differential equation driven by mixed fractional Brownian motion with non-instantaneous impulses have been established, which has not been proposed so far to the best of our knowledge. Finally, we discuss the optimal control problem for a system governed by fractional differential equation in a real Hilbert space. The optimal pair is obtained as the limit of the optimal pair sequence of the unconstrained problem and also, we derive some approximation results, which guarantee the convergence of the numerical method to optimal pair sequence. Several examples have been provided in order to make our theoretical analysis more concrete. In this work, the main techniques used are fixed point theorems, semigroup theory, q-resolvent family, fractional calculus and stochastic analysis theory.

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