ANALYSIS OF IMPULSIVE FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

A THESIS

submitted by

LAKSHMAN MAHTO

for the award of the degree

of

DOCTOR OF PHILOSOPHY

SCHOOL OF BASIC SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY MANDI

MANDI-175001, INDIA

JUNE, 2014

I dedicate this thesis to my parents, my brothers, my sisters,

my teachers

and all those people who help and inspire me, because through you I become whole.

THESIS CERTIFICATE

This is to certify that the thesis titled **"Analysis of Impulsive Fractional Functional Differential Equations"**, submitted by **Lakshman Mahto**, to the Indian Institute of Technology, Mandi, for the award of the degree of **Doctor of Philosophy**, is a bonafide record of the research work done by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

SBS, IIT Mandi, Mandi
Email: abbas@iitmandi.ac.in

Dr. Syed Abbas
(Supervisor)

Declaration by the Research Scholar

I hereby declare that the entire work embodied in this thesis is the result of investigations carried out by me in the School of Basic Sciences, Indian Institute of Technology Mandi, under the supervision of Dr. Syed Abbas, and that it has not been submitted elsewhere for any degree or diploma. In keeping with the general practice, due acknowledgments have been made wherever the work described is based on finding of other investigators.

Mandi 175001

Date:June 13, 2014

Signature:

(LAKSHMAN MAHTO)

Acknowledgments

I would like to express my heartily gratitude and thankfulness to my supervisor, Dr. Syed Abbas. He provided me the great opportunity to pursue my doctoral degree, and has given me constant encouragement and excellent guidance throughout the progress of this research work. Special thanks go to him for encouraging me to think problems in multiple directions.
Next, I am very grateful to Timothy A Gonsalves, Director, IIT Mandi for setting up a good research environment in this Himalayan region. To chairperson and teachers of School of Basic Sciences, IIT Mandi. To my teachers, Dr. K.C. Prasad and Dr. B.B. Bhattacharya who taught me very well in my master and for encouraging me to pursue higher study in mathematics.
I thank to my Doctoral committee members Head SBS IIT Mandi, Dr. Manoj Thakur, Dr. Rajendra K. Ray and Dr. Bharat S. Rajpurohit for their help, support, valuable suggestions and discussions.
I am also grateful to the MHRD India for providing me fellowship to carry out this research work.
I thank to all the staff members of IIT Mandi, especially to Prakash, Shushil, Sumit, J.R. Sharma, C.L. Sharma for their help and support.
I thank to our research group members Jai Prakash, Swati, Sultan for their help, support and valuable discussions.
I immensely express my heartiest thanks to my close friends Vivek, Anamika, Jai Prakash, Syamantak, Manoj, Madhusudan, Krishna, Priyankar who gave a new meaning to my life and, for all their lively discussions and support.
Last but not the least; I would like to convey my special thanks to my dear grandfather, parents, teachers, brothers and sisters who tolerated the onslaughts of time to bring me up to

this position and for their blessings, continuous encouragement and moral support.

(Lakshman Mahto)

ABSTRACT

Keywords: *Fractional operators, Mittage-Leffler function, fixed point methods, $\alpha-$ resolvent family of bounded linear operators, Lyapunov function, Razumikhin technique, fractional Adams-Bashforth-Moulton numerical techniques.*

Impulsive fractional functional differential equations are analyzed and applied to the fields like ecological system, control theory and neural network. Impulsive differential equations, fractional differential equations and functional differential equations, first discussed, then the combination of all these three, as impulsive fractional functional differential equations, are analyzed. Such a generalized differential equation can suitably model, the evolutionary systems that exhibit delay, impulsive effect and anomalous characteristics. After formulating initial value problems for these systems and defining corresponding solution, several theorems on dynamical analysis of solutions of these problems are established and analyzed. In the dynamical analysis, existence, uniqueness, stability, permanence, persistence, numerical simulation of solutions are carried out. Fixed point methods, $\alpha-$ resolvent family of bounded linear operators, Lyapunov function, Razumikhin technique and fractional Adams-Bashforth-Moulton numerical techniques are main tools applied in the analysis of our problems. Finally, several examples and applications are given from motivation point of view.

Contents

Thesis Certificate i

Acknowledgement i

Abstract iii

List of Figures ix

1 Introduction 2

1.1 Impulsive fractional functional differential equations 2

1.1.1 Impulsive differential equations 2

1.1.2 Fractional differential equations 4

1.1.3 Impulsive fractional functional differential equations 4

1.2 Motivation and objective 5

1.3 Problem description 6

1.3.1 P-1: Impulsive fractional differential equation 6

1.3.1.1 Existence problem: 6

1.3.1.2 Uniqueness problem: 7

1.3.2 P-2: Impulsive fractional functional differential equation 7

1.3.2.1 Existence problem: 7

1.3.2.2 Uniqueness problem: 7

1.3.2.3 Stability problem: 8

1.3.3 P-3: Impulsive fractional functional semilinear differential equation 8

1.3.3.1 PC-almost automorphic solution 8

1.3.3.2 Control problem 9

1.3.4 P-4: Fractional phytoplankton model 9

1.3.4.1 Dynamical analysis of fractional phytoplankton problem 10
1.3.4.2 Numerical analysis 10
1.4 Layout of the thesis 10

2 Preliminaries **12**
2.1 Impulsive fractional functional differential equations 12
2.1.1 Impulsive differential equations 12
2.1.2 Fractional differential equations 13
2.1.3 Functional differential equations 15
2.1.4 Impulsive fractional functional differential equations 16
2.2 Some special functions 16
2.3 Fixed point theorems 18
2.4 Lyapunov-like functions and relevant concepts 19
2.5 $\alpha-$ resolvent family of bounded linear operators 20
2.6 Almost automorphic functions 21

3 Impulsive fractional functional differential equations **24**
3.1 Introduction 24
3.2 Impulsive fractional differential equations 26
3.2.1 Existence of solution 26
3.2.2 Uniqueness of solution 31
3.3 Impulsive fractional functional differential equations 32
3.3.1 Existence of solution 33
3.3.2 Uniqueness of solution 36
3.4 Examples 36

4 Stability of impulsive fractional functional differential equations **40**
4.1 Introduction 40
4.1.1 Problem formulation 40
4.1.2 Literature survey 41
4.1.3 Motivation and objective 42
4.2 Preliminaries 43
4.3 Stability Analysis 44

4.4 Examples 49

5 Almost automorphic solutions **52**

5.1 Introduction 52

5.1.1 Problem Description 52

5.1.2 Almost automorphic solution 53

5.2 Existence of PC-almost automorphic solutions 54

5.2.1 Impulsive fractional differential equation 57

5.2.2 Impulsive fractional functional differential equation 61

5.3 Examples 64

5.4 Discussion 69

6 Application to control theory **70**

6.1 Introduction 70

6.1.1 Problem formulation 70

6.1.2 Mathematical control theory 71

6.1.3 Motivation and objective 71

6.2 Preliminaries 71

6.3 Approximate controllability 73

6.4 Existence of optimal control 79

6.5 Example 83

7 Application to ecological systems **86**

7.1 Introduction 86

7.1.1 Model formulation 86

7.1.2 Impulsive phytoplankton models 87

7.1.3 Motivation and objective 89

7.2 Preliminaries 90

7.3 Permanence and persistence 92

7.4 Local stability analysis of equilibrium points 97

7.5 Numerical simulation 99

7.6 Conclusions 102

8 Conclusions **104**

8.1 Summary 104
8.2 Scope of future work: 105

List of Publications **106**

Bibliography **108**