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Analytical Solution of Smoluchowski Equation in Harmonic Oscillator Potential

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Abstract *Non-equilibrium fission has been described by diffusion model. In order to describe the diffusion process analytically, the analytical solution of Smoluchowski equation in harmonic oscillator potential is obtained. This analytical solution is able to describe the probability distribution and the diffusive current with the variable x and t. The results indicate that the probability distribution and the diffusive current are relevant to the initial distribution shape, initial position, and the nuclear temperature T; the time to reach the quasi-stationary state is proportional to friction coefficient β, but is independent of the initial distribution status and the nuclear temperature T. The prerequisites of negative diffusive current are justified. This method provides an approach to describe the diffusion process for fissile process in complicated potentials analytically.*

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Key words: Smoluchowski equation, probability distribution, diffusive current

1 Introduction

Nuclear fission is a complicated nuclear reaction process, and has important value on nuclear power application. The Bohr–Wheeler formula of fission rate is established based on equilibrium statistical theory.[1] On the other hand the induced nuclear fission can also be viewed as a diffusion process over a potential barrier. Kramers derived the formula for fission width in a quasi-stationary approximation by solving the Fokker–Planck equation.[2] Obviously, there really exists a transient time t_q between the beginning of the diffusion process and the attainment of the quasi-stationary. Indeed, the experimental data indicate that the number of light particles evaporated in a heavy-ion induced reaction prior to fission considerably exceeds the expectations based on the statistical model.[3−5] Thus, the research on the transient process in diffusion model is very necessary. In order to explain the experimental data and research nuclear dissipation mechanism, the transient behaviors of the fission rate have been studied by solving the Fokker–Planck equation numerically.[6−8] However, it is very difficult to solve Fokker–Planck equation analytically, so there are only few analytical solutions to some special problems.[9,10] However, if the friction coefficient β is so large that equilibration in momentum space is very rapid, then the Fokker–Planck equation is translated into the Smoluchowski equation, which describes dissipation only in x space. In terms of the Van Kampen method,[11] the Smoluchowski equation can be solved in an analytical approach. But in this method there are some unphysical features as mentioned in Ref. [12]. For instance, there is no simple way to show how the fission rates will be changed with nucleon temperature T, once the fission potential $V(x)$ is fixed.

So far the diffusion model is the only method to describe the non-equilibrium nuclear fission process, so the study on the diffusion model becomes very important. This status prompts us to carry out the present investigation. In order to understand the physical behaviors of the diffusive process in the harmonic oscillator potential, the analytical solution of the Smoluchowski equation has been obtained in this paper. The results show that the diffusive process behaves with a non-equilibrium feature. The probability distribution and the diffusive current are relevant to the initial status and the nuclear temperature T, the time t_q to reach quasi-stationary distribution increases with increasing friction constant β. In some cases the negative diffusive current would occur and be analyzed. The analytical solution can clearly reflect the influence of every physical quantity on the diffusion process with the simple form. This method provides an approach to make the analytical description of the fissile diffusion process in even complicated potentials.

The analytical probability distributions $P(x,t)$ of the Smoluchowski equation in harmonic oscillator potential are presented in Sec. 2. In Sec. 3 the analytical diffusive current $J(x,t)$ and the prerequisite of the negative current have been given. In Sec. 4, the influence of the physical quantity on the diffusive process is discussed. The analysis and prospects are elaborated in the last section.

2 Analytical Solution of Smoluchowski Equation

Induced nuclear fission can be viewed as a diffusion process over the fission barrier, and can be described by means of the Fokker–Planck equation, which contains the fission variable and its canonically conjugate momentum.

Chandrasekhar reviewed the transition from the Fokker–Planck equation to the Smoluchowski equation,[13] which describes dissipation only in the x space, when the friction coefficient β is so large that equilibration in momentum space is very rapid. The Smoluchowski equation reads

$$\frac{\partial}{\partial t}P(x,t)=\frac{1}{\beta}\frac{\partial}{\partial x}[K(x)P(x,t)]+\frac{1}{\beta^2}\varepsilon\frac{\partial^2}{\partial x^2}P(x,t)\,. \quad (1)$$

Here, $P(x,t)$ is the probability distribution in x space, x is the fission variable; $K(x)\equiv(1/\mu)(\mathrm{d}V(x)/\mathrm{d}x)$, where $V(x)$ stands for the fission potential; $\varepsilon\equiv\beta T/\mu$, T is the nuclear temperature and μ refers to reduced mass, respectively.[11,12] Since the Smoluchowski equation (1) has a simple scaling property as β is changed, to apply this fact, a new "time" $\tau=t/\beta\hbar^2=c^2t/\beta(\hbar c)^2$ is employed in this paper. Thus equation (1) depends only on x and τ, and hence has a common form as

$$\frac{\partial P(x,\tau)}{\partial\tau}=-\frac{\partial}{\partial x}[C(x)P(x,\tau)]+D\frac{\partial^2}{\partial x^2}P(x,\tau)\,, \quad (2)$$

where the drifting coefficient is $C(x)=-(\hbar\omega)^2x$. The harmonic oscillator potential in new scaling reads

$$V(x)=\frac{\mu c^2(\hbar\omega)^2}{2(\hbar c)^2}x^2\,. \quad (3)$$

Here, ω is the harmonic-oscillator frequency, $c=3\times10^{23}\ \mathrm{fm\cdot s^{-1}}$ is light velocity and $\hbar c=197.33\ \mathrm{MeV\cdot fm}$, respectively. The diffusion coefficient $D=(\hbar c)^2T/\mu c^2$ depends on $T/\mu c^2$, which is independent of x, so we no longer discuss the influence of μ in this paper because μ is always accompanying with the temperature T.

The analytical solution of Eq. (2) can be written in the form as

$$P(x,\tau)=\frac{1}{\sqrt{2\pi D\sigma(\tau)}}\exp\Big\{-\frac{[x-\chi(\tau)]^2}{2D\sigma(\tau)}\Big\}\,. \quad (4)$$

Obviously, the normalization condition $\int_{-\infty}^{\infty}P(x,\tau)\mathrm{d}x=1$ always holds. Substituting Eq. (4) into Eq. (2), three equations for different x powers can be issued.

The coefficients of x^0 is obtained by

$$\frac{\chi^2(\tau)}{2D\sigma(\tau)}\Big[\frac{\mathrm{d}\sigma(\tau)}{\mathrm{d}\tau}-2\Big]-\frac{1}{2}\frac{\mathrm{d}\sigma(\tau)}{\mathrm{d}\tau}-\frac{\chi(\tau)}{D}\frac{\mathrm{d}\chi(x)}{\mathrm{d}x}-(\hbar\omega)^2\sigma(\tau)+1=0\,. \quad (5)$$

The coefficients of x^1 is obtained by

$$\sigma(\tau)\frac{\mathrm{d}\chi(\tau)}{\mathrm{d}\tau}-\chi(\tau)\frac{\mathrm{d}\sigma(\tau)}{\mathrm{d}\tau}-(\hbar\omega)^2\sigma(\tau)\chi(\tau)+2\chi(\tau)=0\,. \quad (6)$$

The coefficients of x^2 gives the equation of $\sigma(\tau)$,

$$\frac{\mathrm{d}\sigma(\tau)}{\mathrm{d}\tau}=-2(\hbar\omega)^2\sigma(\tau)+2\,. \quad (7)$$

Substituting Eq. (7) into Eq. (6), the equation of $\chi(\tau)$ is given by

$$\frac{\mathrm{d}\chi(\tau)}{\mathrm{d}\tau}=-(\hbar\omega)^2\chi(\tau)\,. \quad (8)$$

Substituting Eqs. (7) and (8) into Eq. (5), one can find that equation (5) is naturally satisfied. This fact implies the correctness of the analytical representation of Eq. (4).

The initial probability distributions is taken as a normalized Gaussian form,

$$P(x,\tau=0)=\frac{1}{\sqrt{2\pi}\sigma_0}\exp\Big\{-\frac{(x-x_0)^2}{2\sigma_0^2}\Big\}\,, \quad (9)$$

where the position of the peak at $x=x_0$, as shown in Fig. 1.

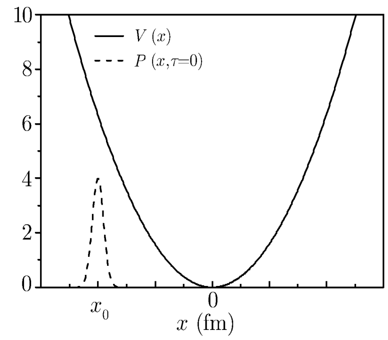

Fig. 1 Harmonic oscillator potential and the initial distributions.

With the initial condition of Eq. (9), the analytical solution of Eqs. (7) and (8) can be easily obtained as

$$\sigma(\tau)=\frac{\sigma_0^2}{D}\,\mathrm{e}^{-2(\hbar\omega)^2\tau}+\frac{1}{(\hbar\omega)^2}(1-\mathrm{e}^{-2(\hbar\omega)^2\tau})\,, \quad (10)$$

$$\chi(\tau)=x_0\,\mathrm{e}^{-(\hbar\omega)^2\tau}\,. \quad (11)$$

Obviously,

$$\sigma(0)=\frac{\sigma_0^2}{D},\quad \sigma(\infty)=\frac{1}{(\hbar\omega)^2}\,,$$

$$\chi(0)=x_0,\quad \chi(\infty)=0\,.$$

When $\tau\to\infty$, the quasi-stationary probability distributions is given by

$$P(x,\tau\to\infty)=\frac{\hbar\omega}{\sqrt{2\pi D}}\exp\Big\{-\frac{(\hbar\omega)^2}{2D}x^2\Big\}=\frac{\hbar\omega}{\hbar c}\sqrt{\frac{\mu c^2}{2\pi T}}\exp\Big\{-\frac{(\hbar\omega)^2}{2(\hbar c)^2}\frac{\mu c^2}{T}x^2\Big\}\,. \quad (12)$$

3 Diffusive Current in Harmonic Oscillator Potential

If the Smoluchowski equation is written into the continuity equation form as

$$\frac{\partial}{\partial t}P(x,t)+\nabla\cdot J(x,t)=0\,. \quad (13)$$

It is convenient to express the diffusive current with x and τ.

$$J(x,\tau) = -\frac{(\hbar c)^2}{\mu c^2}\frac{\mathrm{d}V(x)}{\mathrm{d}x}P(x,\tau) - \frac{(\hbar c)^2}{\mu c^2}T\frac{\mathrm{d}P(x,\tau)}{\mathrm{d}x}\,. \tag{14}$$

Inserting Eqs. (4), (10), and (11) into Eq. (14), the analytical expression of the diffusive current is obtained by

$$J(x,\tau) = \frac{x-\chi(\tau)-(\hbar\omega)^2\sigma(\tau)x}{\sigma(\tau)}P(x,\tau)\,. \tag{15}$$

Obviously, for $\tau \to 0$ the diffusive current has the form as

$$J(x,\tau\approx 0) = \frac{[D-(\hbar\omega)^2\sigma_0^2]x - Dx_0}{\sigma_0^2}P(x,\tau\approx 0)\,. \tag{16}$$

For sufficiently large τ the diffusive current should vanish,

$$J(x,\tau\to\infty) = 0\,. \tag{17}$$

Because $P(x,\tau)$ and $\sigma(\tau)$ always have positive values for any x or τ, hence, it is possible to have the negative current $J(x,\tau) < 0$, if the inequality

$$[1-(\hbar\omega)^2\sigma(\tau)]x < \chi(\tau) \tag{18}$$

holds. Substituting Eqs. (10) and (11) into Eq. (18), the condition (18) becomes

$$\Big[1-\frac{(\hbar\omega)^2}{D}\sigma_0^2\Big]x < x_0\,\mathrm{e}^{(\hbar\omega)^2\tau}\,. \tag{19}$$

When τ or $|x|$ is sufficiently large, equation (15) implies $P(x,\tau)\to 0$ and $J(x,\tau)\to 0$, so the negative current is not easy to be observed. Only at a small value of τ can the negative current be observed. Without losing generality, $x_0 < 0$ is taken, which means that the probability locates at the left side of the potential well.

i) When $[1-(\hbar\omega)^2\sigma_0^2/D] > 0$, or $D > (\hbar\omega)^2 \times\sigma_0^2$ is satisfied, there is negative current at $x < x_0\,\mathrm{e}^{(\hbar\omega)^2\tau}/[1-(\hbar\omega)^2\sigma_0^2/D]$ region. In this region, the diffusive behavior is quicker than the drifting velocity. The peak of the initial distribution with the Gaussian form suddenly collapses, at the same time its bottom extends to both sides. We call this phenomenon as "Gaussian Collapses". This case is corresponding to the high nuclear temperature and narrow initial configuration.

ii) When $[1-(\hbar\omega)^2\sigma_0^2/D] < 0$, or $D < (\hbar\omega)^2 \times\sigma_0^2$ is satisfied, there is negative current at $x > -x_0\,\mathrm{e}^{(\hbar\omega)^2\tau}/[(\hbar\omega)^2\sigma_0^2/D-1]$ region. In this region, the width of the quasi-equilibrium distribution is narrower than the width of the initial distribution, so the initial distribution shrinks towards the center of the quasi-equilibrium distribution. We call this phenomenon as "Gaussian Shrinks". This case is corresponding to the low nuclear temperature and very wide initial configuration. But for this configuration, the probability density commonly is very small at $x > -x_0$, so the negative current is not easily observed.

4 Relationship of Probability Distribution and Diffusive Current with Initial Distribution and Nuclear Temperature

According to the analytical expressions of the probability distribution and the diffusive current, "time" τ_q for attaining quasi-stationary distribution can be decided by the factor $\exp\{-(\hbar\omega)^2\tau_q\}\approx 0.01$, which yields

$$\tau_q \approx \frac{4.6}{(\hbar\omega)^2}\,. \tag{20}$$

Translating into the real time

$$t_q \approx 4.6\frac{\beta}{\omega^2}\,. \tag{21}$$

As mentioned above, Smoluchowski equation only suits for large friction, namely $\beta \geq 5\times 10^{21}$ s^{-1}. If taking $\hbar\omega \approx 1$ MeV, then $t_q \geq 1\times 10^{-21}$ s. Equation (21) indicates that t_q is proportional to β and ω^{-2}, but independent of T, as well as the initial distribution. The larger β is, the slower the diffusion process is. Contrarily, the larger $\hbar\omega$ is, the quicker the diffusion process is. In this paper, we take $\hbar\omega = 1.0$ MeV and $\mu = 63.5$ m, where m is the nucleon mass, as same as used in Ref. [12].

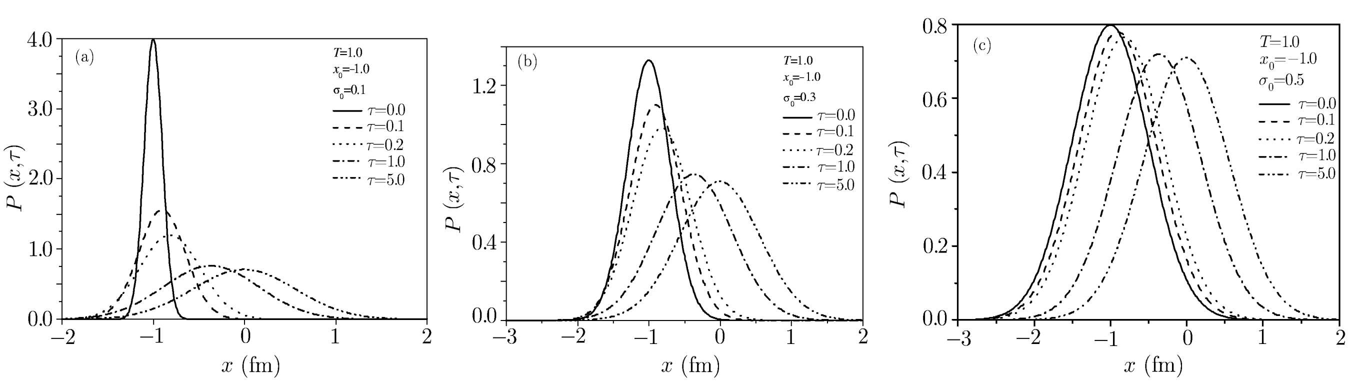

Fig. 2 The relationship of the probability distribution with the width of the initial widths.

For different initial distribution widths $\sigma_0 = 0.1, 0.3, 0.5$ fm, respectively, with $T = 1.0$ MeV, and $x_0 = -1.0$ fm, the evolution of the probability distributions $P(x,\tau)$ with time is shown in Fig. 2, from which we can see that the narrower σ_0 is, the quicker the evolution is. Finally, they reach the quasi-stationary state simultaneously.

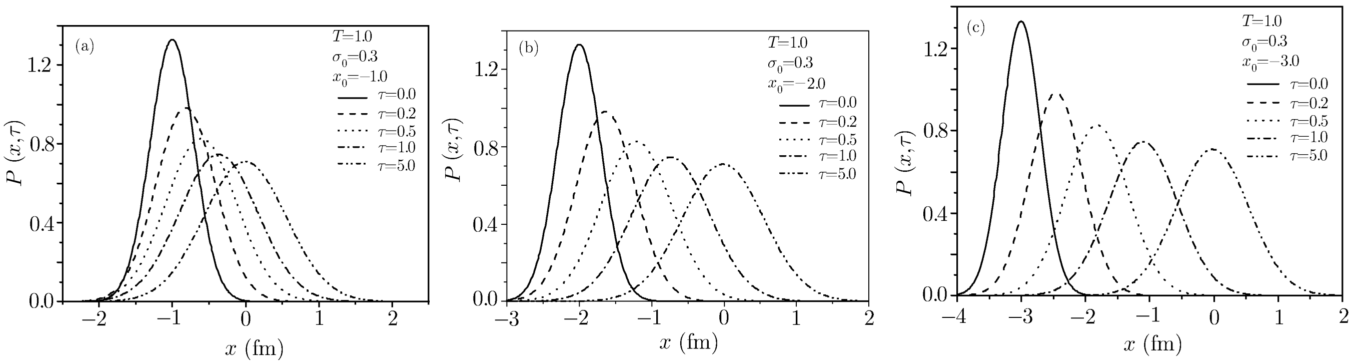

Fig. 3 The relationship of the probability distribution with the position of the initial position.

Figure 3 shows the evolution of the probability distributions $P(x,\tau)$ for different initial position $x_0 = -1.0$, -2.0, -3.0 fm, respectively, with $T = 1.0$ MeV and $\sigma_0 = 0.3$ fm. The results show that the lager the value $|x_0|$ is, the quicker the drift velocity is. The shape of $P(x,\tau)$ is the same at any time, although x_0 is different from one another.

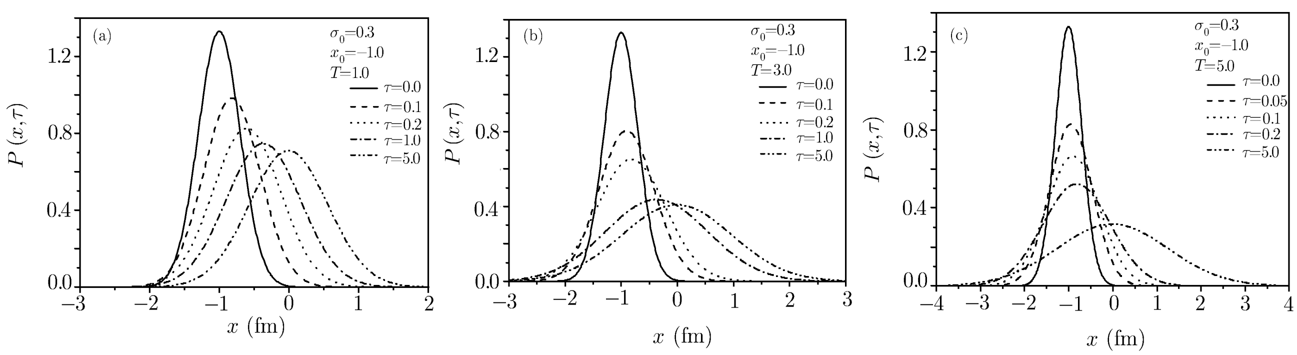

Fig. 4 The relationship of the probability with the nuclear temperature.

In Fig. 4 the relationship of $P(x,\tau)$ vs. x with the nuclear temperature $T = 1.0, 3.0, 5.0$ MeV, respectively, with $x_0 = -1.0$ fm and $\sigma_0 = 0.3$ fm is given. The results indicate that the higher T is, the wider the quasi-stationary distribution is.

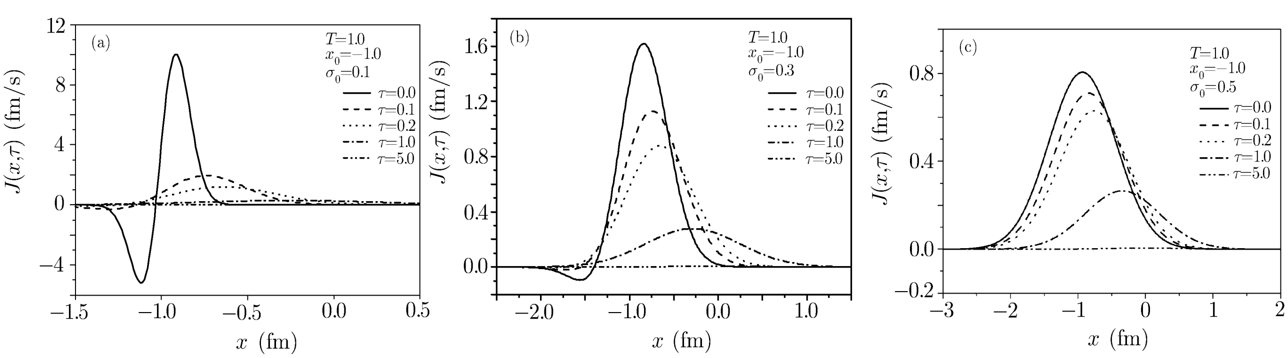

Fig. 5 The relationship of the diffusive current with the width of the initial distribution.

Figure 5 shows the diffusive current $J(x,\tau)$ for initial width $\sigma_0 = 0.1, 0.3, 0.5$ fm, respectively, with $T = 1.0$ MeV, $x_0 = -1.0$ fm. In quasi-stationary state $J(x,\tau \to \infty)$ tends to zero. The wider σ_0 is, the slower $J(x,\tau)$ is at each time. Since the widths are small in Figs. 5(a) and 5(b), so the negative current occurs at $x < x_0$ region. This is the "Gaussian Collapses" phenomenon.

The relationship of diffusive current $J(x,\tau)$ with the initial position are shown in Fig. 6 for $x_0 = -1.0$, -2.0, -3.0 fm, respectively, with $T = 1.0$ MeV and $\sigma_0 = 0.3$ fm. The further away it is from the well, the quicker the diffusive current is.

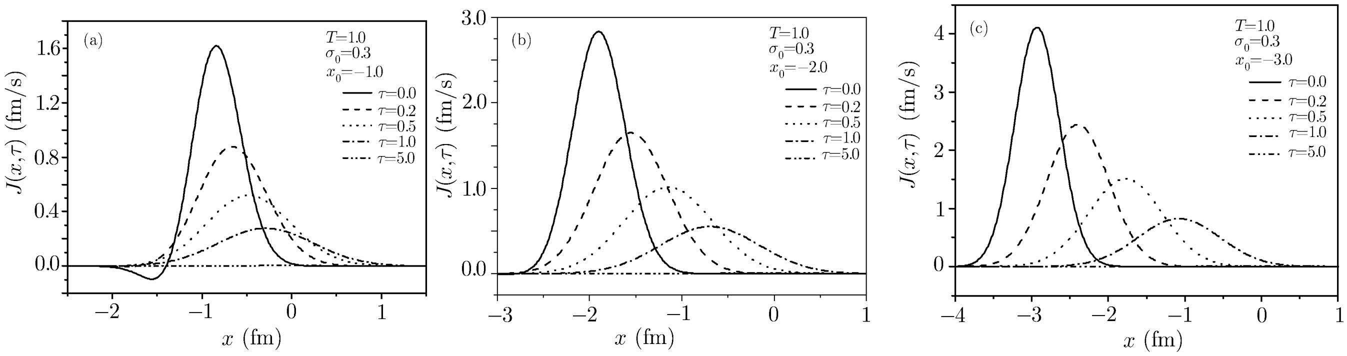

Fig. 6 The relationship of the diffusive current with the position of the initial distribution.

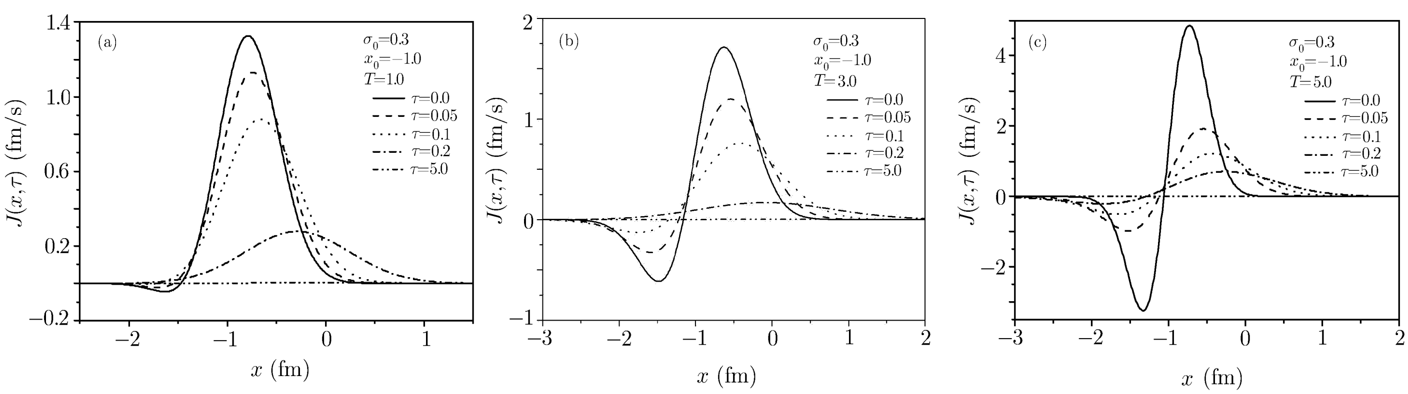

Fig. 7 The relationship of the diffusive current with the nuclear temperature.

Figure 7 shows the relationship of diffusive current $J(x,\tau)$ with the temperature for $T = 1.0, 3.0, 5.0$ MeV, respectively, with $x_0 = -1.0$ fm and $\sigma_0 = 0.3$ fm. The higher the temperature is, the rapider the diffusive current is, and the more obvious the negative current is. This phenomenon is in accordance with the discussion of Sec. 3.

5 Summary

In summary, the approach to get the analytical solution of Smoluchowski equation in one dimension at harmonic oscillator potential is expounded in detail. We analyze the laws of the probability distribution and the diffusive current at different initial conditions and the nuclear temperatures. Of course, the influence from the reduced mass, as well as the harmonic-oscillator frequency is obvious. The former is always accompanied with the temperature, the change of the reduced mass being corresponding to the change of the temperature, while the latter is always accompanied with the "time" τ, with the increasing of the frequency the diffusion process being more quicker. The calculated results indicate that the time to reach the quasi-stationary state is proportional to β and ω^{-2}. The evolution of probability distributions and the diffusive current have been shown in the figures, with different initial distribution and nuclear temperature T. There is "Gaussian Collapses" phenomenon, when the nuclear temperature T is high and the initial distribution width σ_0 is narrow. Also there is "Gauss Shrinks" phenomenon, when T is low and σ_0 is wide. The analytical representation of Smoluchowski equation is simple and can reflect clearly the influence on the diffusive process with every physical quantity. This approach to solve the Smoluchowski equation analytically may go a step further for more complicated potentials to study the fission processes in the future.

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