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Time-dependent scattering theory for infinite delta function potentials in one dimension

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Existence of the Møller wave operators is proved for a system of n quantum mechanical particles interacting through infinite delta function potentials in one dimension.

INTRODUCTION

There has been some discussion in the literature of n-body quantum-mechanical systems in one dimension with interaction between the particles provided by "infinite delta function potentials" (see for example, Refs. 1, 2). Such systems have extremely simple properties and are worth study for this very reason. Scattering theory for these systems (and others) has been analyzed in using time-independent methods. The purpose of the present note is merely to point out that the corresponding time-dependent version of scattering theory can be given for these systems: the Møller wave operators exist and the S matrix is unitary. The proofs are very simple, and in the present author's opinion some salient features of the theory stand out more clearly in the time-dependent version.

1. THE HAMILTONIAN

To reinforce the reader's intuition, we consider first the case of one particle in a potential. We wish to make sense of the operator

$$H=-\frac{1}{2m}\frac{d^2}{dx^2}+\infty\delta(x) \tag{1}$$

on the Hilbert space $L^2(\mathbf{R})$. Intuitively, the idea to be exploited is that the particle cannot pass through the origin. Thus it is natural to view $L^2(\mathbf{R})$ as the direct sum of $L^2(-\infty,0)$ and $L^2(0,\infty)$. On each of the latter spaces, H should act like the free Hamiltonian $H_0=-(1/2m)d^2/dx^2$ *with zero boundary conditions at* $x=0$. To apply this latter operator to a function $\psi\in L^2(0,\infty)$ means: Take the Fourier *sine* transform $\hat{\psi}$ of ψ, multiply $\hat{\psi}$ by $k^2/2m$, and take the inverse Fourier sine transform of the result. This is equivalent to the following: Take the *odd* extension of ψ to obtain a function ψ_{odd} in $L^2(\mathbf{R})$, apply the free Hamiltonian H_0 [considered as an operator on $L^2(R)$ in the usual way] to ψ_{odd}, and truncate $H_0\psi_{\text{odd}}$ to obtain a function in $L^2(0,\infty)$. By this kind of analysis, we arrive at the following description of the operator H: Define operators $P\pm$ and A on $L^2(\mathbf{R})$ as follows:

$$(P_+\psi)(x)=\begin{cases}\psi(x) & (x>0),\\ 0 & (x<0),\end{cases} \tag{2}$$

$$(P_-\psi)(x)=\begin{cases}0 & (x>0),\\ \psi(x) & (x<0),\end{cases} \tag{3}$$

$$(A\psi)(x)=\psi(x)-\psi(-x). \tag{4}$$

Then

$$H=P_+H_0AP_++P_-H_0AP_-. \tag{5}$$

Naturally the situation we have just discussed is equivalent to a two-body problem in which the center-of-mass coordinate has been separated out. If we interpret x as the relative coordinate x_1-x_2 of this two-body problem, then the conditions $x>0$ and $x<0$ correspond to $x_1>x_2$ and $x_2>x_1$. The definition of the n-body Hamiltonian

$$H=\sum_{i=1}^{n}-\frac{1}{2m_i}\frac{d^2}{dx_i^2}+\sum_{i<j}\infty\delta(x_i-x_j) \tag{6}$$

is given in terms of operators projecting on subspaces of $L^2(\mathbf{R}^n)$ in which a certain order $x_{i_1}>x_{i_2}>\cdots>x_{i_n}$ prevails among the particle coordinates.

Let S_n be the symmetric group on n elements, and for each $\pi\in S_n$ let S_π be the following subset of $\mathbf{R}^n$:

$$S_\pi=\{(x_1,\ldots,x_n)\mid x_{\pi1}>x_{\pi2}>\cdots>x_{\pi n}\}. \tag{7}$$

Let P_π be the following projection in $L^2(\mathbf{R}^n)$:

$$(P_\pi\psi)(x_1,\ldots,x_n)=\begin{cases}\psi(x_1,\ldots,x_n) & \text{if } (x_1,\ldots,x_n)\in S_\pi,\\ 0 & \text{otherwise.}\end{cases} \tag{8}$$

Let $\sigma(\pi)$ be the parity of the permutation π, and let A be the following operator on $L^2(\mathbf{R}^n)$:

$$(A\psi)(x_1,\ldots,x_n)=\sum_{\pi\in S_n}\sigma(\pi)\psi(x_{\pi1},x_{\pi2},\ldots,x_{\pi n}). \tag{9}$$

It is easy to verify that

$$P_\pi AP_\pi=P_\pi,\quad AP_\pi A=A. \tag{10}$$

Writing

$$H_0=\sum_{i=1}^{n}-\frac{1}{2m_i}\frac{d^2}{dx_i^2}, \tag{11}$$

the definition of H is now simply

$$H=\sum_{\pi\in S_n}P_\pi H_0AP_\pi. \tag{12}$$

We have correspondingly

$$e^{iHt}=\sum_{\pi\in S_n}P_\pi\exp(iH_0t)AP_\pi. \tag{13}$$

It should be noted that H and H_0 both commute with A.

2. SCATTERING THEORY

We will establish existence of the Møller wave operators

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$$W_\pm = \text{s-}\lim_{t\to\pm\infty} \exp(iHt)\exp(-iH_0t). \tag{14}$$

We do not attempt the usual proof in which $\exp(iHt) \times\exp(-iH_0t)$ is differentiated with respect to t, because H and H_0 do not have the same domain. Instead we proceed as follows: Let F: $L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$ denote the operation of Fourier transformation in all the variables. Let C_t be the operator defined for $t\neq 0$ by

$$(C_t\psi)(x_1,\ldots,x_n) = \frac{(m_1,\ldots,m_n)^{1/2}}{(it)^{n/2}} \exp\left(i\sum_{i=1}^n \frac{m_i x_i^2}{2t}\right) \times (F\psi)\left(\frac{m_1x_1}{t},\ldots,\frac{m_nx_n}{t}\right). \tag{15}$$

Then we have for any $\psi\in L^2(\mathbf{R}^n)$ (see Ref. 3)

$$\lim_{t\to\pm\infty} \|\exp(-iH_0t)\psi - C_t\psi\| = 0. \tag{16}$$

Now

$$\begin{aligned}\exp(iHt)\exp(-iH_0t) &= \sum_{\pi\in S_n} P_\pi A\exp(iH_0t)P_\pi\exp(-iH_0t)\\ &= \sum_{\pi\in S_n} P_\pi A B_\pi(t),\end{aligned} \tag{17}$$

where

$$B_\pi(t) = \exp(iH_0t)P_\pi\exp(-iH_0t). \tag{18}$$

Let $\varphi,\psi\in L^2(\mathbf{R}^n)$. By (16) we have (in the sense that the difference of the two sides goes to zero)

$$\begin{aligned}(\varphi, B_\pi(t)\psi) &= (\exp(-iH_0t)\varphi, P_\pi\exp(-H_0t)\psi) \underset{t\to\pm\infty}{\to} (C_t\varphi, P_\pi C_t\psi)\\ &= (m_1\ldots m_n/|t|^n)\int_{R^n} (F\varphi)(m_1x_1/t,\ldots,m_nx_n/t)\\ &\quad\times P_\pi(F\psi)(m_1x_1/t,\ldots,m_nx_n/t)dx_1\ldots dx.\end{aligned} \tag{19}$$

Making the change of variables $k_1 = m_1x_1/t,\ldots, k_n = m_nx_n/t$, we obtain

$$\begin{aligned}(\varphi, B_\pi(t)\psi) \underset{t\to\pm\infty}{\to} \int \overline{(F\varphi)(k_1,\ldots,k_n)}\, P_{\pm\pi}\\ \times(F\psi)(k_1,\ldots,k_n)dk_1\ldots dk_n,\end{aligned} \tag{20}$$

where $+\pi=\pi$ and $-\pi$ is defined as follows: If

$$\pi = \begin{pmatrix}1 & 2 & \cdots & n\\ \pi 1 & \pi 2 & \cdots & \pi n\end{pmatrix},$$

then

$$-\pi = \begin{pmatrix}1 & 2 & \cdots & n\\ \pi n & \pi(n-1) & \cdots & \pi 1\end{pmatrix}.$$

Equation (20) can be rewritten as follows:

$$B_\pi(t) \underset{t\to\pm\infty}{\xrightarrow{\text{weak}}} F^{-1}P_{\pm\pi}F. \tag{21}$$

Thus

$$\exp(iHt)\exp(-iH_0t) \underset{t\to\pm\infty}{\xrightarrow{\text{weak}}} \sum_{\pi\in S_n} P_\pi A F^{-1} P_{\pm\pi} F \equiv W_\pm. \tag{22}$$

Now using (10) and the facts that $F^* = F^{-1}$ and F commutes with A, we have

$$\begin{aligned}W_\pm^* W_\pm &= \sum_{\pi,\pi'\in S_n} F^{-1}P_{\pm\pi'}FAP_{\pi'}P_\pi AF^{-1}P_{\pm\pi}F\\ &= \sum_{\pi\in S_n} F^{-1}P_{\pm\pi}FAP_\pi AF^{-1}P_{\pm\pi}F\\ &= \sum_{\pi\in S_n} F^{-1}P_{\pm\pi}FAF^{-1}P_{\pm\pi}F\\ &= \sum_{\pi\in S_n} F^{-1}P_{\pm\pi}F = I.\end{aligned} \tag{23}$$

Since the unitary operator $\exp(iHt)\exp(-iH_0t)$ converges weakly to $W_\pm$, and since by (23) $W_\pm$ is an isometry, it follows that the convergence in (22) is in fact strong. Thus existence of a scattering theory in the time-dependent sense is established. A calculation similar to (23) shows that $W_\pm W_\pm^* = I$ so that, as expected in a theory with no bound states, the Møller wave operators are unitary. The S matrix $W_\pm^* W_-$ is then obviously unitary. A calculation similar to (23) yields

$$\begin{aligned}S = W_+^*W_- &= \sum_{\pi\in S_n} F^{-1}P_\pi AP_{-\pi}F\\ &= \sum_{\pi\in S_n} F^{-1}P_{-\pi}AP_\pi F.\end{aligned} \tag{24}$$

Viewed in momentum space, the S matrix has the extremely simple form

$$FSF^{-1} = \sum_{\pi\in S_n} P_{-\pi}AP_\pi. \tag{25}$$

This equation states that a contribution to the incoming wave function at given momenta $k_{\pi 1} > k_{\pi 2} > \cdots > k_{\pi n}$ (particle πi having momentum $k_{\pi i}$ will produce a contribution to the outgoing wave function in which particle $\pi 1$ has momentum $k_{\pi n}$, particle $\pi 2$ has a momentum $k\pi(n-1)$, etc. This is the expected result: The scattering at given $k_{\pi 1} > k_{\pi 2} > \cdots > k_{\pi n}$ exactly mimics the behavior of classical point particles on a line undergoing elastic collisions, in which they exchange momenta.

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